Eksamen R2
Høst 2020
Løsningsforslag

DEL 1

Oppgave 1 (4 poeng)

a) $f(x) = \sin(2x) + \pi$

$f'(x) = \cos(2x) \cdot 2 = 2\cos(2x)$

b) $g(x) = x \cdot \cos^2(x)$

$g'(x) = \cos^2 x + x \cdot 2\cos x(-\sin x) = \cos x(\cos x - 2x \cdot \sin x)$

Oppgave 2 (6 poeng)

a)

$$\int \left(\cos x + \frac{1}{x} \right) \, dx = \sin x + \ln |x| + C$$

b)

$$\begin{aligned}
\int x \cdot e^{2x} \, dx = \\
u = x, u' = 1 \\
v = \frac{1}{2}e^{2x}, v' = e^{2x} \\
&= \frac{1}{2}x \cdot e^{2x} - \int \frac{1}{2}e^{2x} \, dx \\
&= \frac{1}{2}x \cdot e^{2x} - \frac{1}{2} \cdot \frac{1}{2} \cdot e^{2x} + C \\
&= \frac{1}{2}e^{2x}\left(x - \frac{1}{2} \right) + C
\end{aligned}$$

c)

$$\begin{aligned}
\int \frac{2x-2}{x^2-2x-3} \, dx &= \int \frac{2(x-1)}{(x-3)(x+1)} \\
\text{Mellomregning :} \frac{2x-2}{(x-3)(x+1)} &= \frac{A}{(x-3)} + \frac{B}{(x+1)} \\
2x - 2 &= A(x+1) + B(x-3) \\
x = 3 : 6 - 2 &= 4A \Rightarrow A = 1 \\
x = -1 : -2 - 2 &= -4B \Rightarrow B = 1
\end{aligned}$$

$$\begin{aligned}
\int \frac{2(x-1)}{(x-3)(x+1)} &= \int \frac{1}{(x-3)} + \frac{1}{(x+1)} \\
&= \ln |(x-3)| + \ln |(x+1)| + C \\
&= \ln |(x-3)(x+1)| + C
\end{aligned}$$

Oppgave 3 (4 poeng)

a)

$$\cos(2x) = \frac{1}{2}$$
$$2x = \frac{\pi}{3} + n \cdot 2\pi \vee 2x = -\frac{\pi}{6} + n \cdot 2\pi$$
$$x = \frac{\pi}{6} + n \cdot \pi \vee x = -\frac{\pi}{6} + n \cdot \pi$$
$$x = \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$$

b)

$$\sqrt{3}\sin x - \cos x = 1$$
$$A = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$$
$$2(\sin x \cdot \frac{\sqrt{3}}{2} - \cos x \cdot \frac{1}{2}) = 1$$
$$2(\sin x \cdot \cos\frac{\pi}{6} - \cos x \cdot \sin\frac{\pi}{6}) = 1$$
$$2\sin(x - \frac{\pi}{6}) = 1$$
$$\sin(x - \frac{\pi}{6}) = \frac{1}{2}$$
$$x - \frac{\pi}{6} = \frac{\pi}{6} + n \cdot 2\pi \vee x - \frac{\pi}{6} = \frac{5\pi}{6} + n \cdot 2\pi$$
$$x = \frac{2\pi}{6} + n \cdot 2\pi \vee x = \frac{6\pi}{6} + n \cdot 2\pi$$
$$x = \frac{\pi}{3} + n \cdot 2\pi \vee x = \pi + n \cdot 2\pi$$
$$x = \left\{\frac{\pi}{3}, \pi\right\}$$

Oppgave 4 (6 poeng)

$$\alpha : 2x - 3y + z = 13$$

a) $A(4, -2, -1) : 2 \cdot 4 - 3 \cdot (-2) + (-1) = 8 + 6 - 1 = 13$, altså ligger A i planet.

$P(1, 2, 3) : 2 \cdot 1 - 3 \cdot 2 + 3 = 2 - 3 + 3 \neq 13$, altså ligger IKKE P i planet.

b) Normalvektor til planet er retningsvektoren til linja siden linja står normalt på planet : $\vec{r_l} = [2, -3, 1]$.

$$l = \begin{cases} x = 1 + 2t \\ y = 2 - 3t \\ z = 3 + t \end{cases}$$

Skjæringspunkt mellom l og α :
Setter koordinatene til linje inn i planet :

$$\begin{aligned} 2(1 + 2t) - 3(2 - 3t) + (3 + t) &= 13 \\ 2 + 4t - 6 + 9t + 3 + t &= 13 \\ 14t &= 14 \\ t &= 1 \\ &S : (1 + 2, 2 - 3, 3 + 1) \\ &S : (3, -1, 4) \end{aligned}$$

c)

$$\begin{aligned} \overrightarrow{SA} &= [4 - 3, -2 - (-1), -1 - 4] \\ &= [1, -1, -5] \\ |\overrightarrow{SA}| &= |[1, -1, -5]| \\ &= \sqrt{1^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{27} = 3\sqrt{3} \\ \overrightarrow{SP} &= [1 - 3, 2 - (-1), 3 - 4] \\ &= [-2, 3, -1] \\ |\overrightarrow{SP}| &= |[-2, 3, -1]| \\ &= \sqrt{(-2)^2 + 3^2 + (-1)^2} \\ &= \sqrt{4 + 9 + 1} = \sqrt{14} \\ V &= \frac{1}{3} \cdot \pi \cdot \sqrt{27}^2 \cdot \sqrt{14} \\ &= 9\sqrt{14}\pi \end{aligned}$$

Oppgave 5 (4 poeng)

$$f(x) = a \cdot \cos\left(\frac{\pi}{2}x - b\right) + 2$$

a)

$$\begin{aligned}
f'(x) &= a \cdot \left(- \sin\left(\frac{\pi}{2}x - b\right) \cdot \frac{\pi}{2}\right) \\
&= -a \cdot \frac{\pi}{2} \sin\left(\frac{\pi}{2}x - b\right) \\
f'(3) &= 0 \\
-a \cdot \frac{\pi}{2} \sin\left(\frac{3\pi}{2} - b\right) &= 0 \\
\sin\left(\frac{3\pi}{2} - b\right) &= 0 \\
\frac{3\pi}{2} - b = n \cdot \pi & \\
b &= \frac{3\pi}{2}
\end{aligned}$$

$$\begin{aligned}
f(x) &= a \cdot \cos\left(\frac{\pi}{2}x - b\right) + 2 \\
f(3) &= 5 \\
5 &= a \cdot \cos\left(\frac{3\pi}{2} - \frac{3\pi}{2}\right) + 2 \\
a \cdot \cos 0 + 2 &= 5 \\
a &= 3
\end{aligned}$$

Oppgave 6 (4 poeng)

a)

$$
\begin{aligned}
y'' - 2y' + y &= 0 \\
y &= A \cdot e^x + B \cdot x \cdot e^x \\
&= e^x(A + Bx) \\
y' &= A \cdot e^x + B \cdot e^x + B \cdot x \cdot e^x \\
&= e^x(A + B + Bx) \\
y'' &= A \cdot e^x + B \cdot e^x + B \cdot e^x + B \cdot x \cdot e^x \\
&= e^x(A + 2B + Bx) \\
y'' - 2y' + y &= 0 \\
e^x(A + 2B + Bx) - 2e^x(A + B + Bx) + e^x(A + Bx) &= 0 \\
e^x(A - 2A + A + 2B - 2B + Bx - 2Bx + Bx) &= 0
\end{aligned}
$$

b)

$$
\begin{aligned}
y(0) &= 3 \\
y &= e^x(A + B \cdot x) \\
3 &= e^0(A + B \cdot 0) \\
A &= 3
\end{aligned}
$$

$$
\begin{aligned}
y'(0) &= 5 \\
y' &= e^x(A + B + Bx) \\
5 &= e^0(3 + B + B \cdot 0) \\
B &= 5 - 3 = 2
\end{aligned}
$$

$$
y = e^x(3 + 2x)
$$

Oppgave 7 (5 poeng)

$$S(x) = 1 + \frac{3}{2}\cos x + \frac{9}{4}\cos^2 x + ... + \left(\frac{3}{2}\cos x\right)^{n-1} + ...$$

$k = \left(\frac{3}{2}\cos x\right)$

a) $x = \frac{\pi}{6} \Rightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$k = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4} > 1$, konvergerer ikke. $\quad (\frac{3\sqrt{3}}{4})^2 = \frac{27}{16}$

$x = \frac{\pi}{3} \Rightarrow \cos\frac{\pi}{3} = \frac{1}{2}$

$k = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} < 1$, konvergerer.

b) $x = \frac{\pi}{3} \Rightarrow k = \frac{3}{4}$

$S(x) = \frac{a_1}{1-k} = \frac{1}{1-\frac{3}{4}} = \frac{1}{\frac{1}{4}} = 4$

c) Konvergensområde :

$$-1 < k < 1$$
$$-1 < \frac{3}{2}\cos x < 1$$
$$-\frac{2}{3} < \cos x < \frac{2}{3}$$

$$S(x) = r$$
$$\frac{1}{1-k} = r, \text{ når } k \in \langle -1, 1 \rangle$$
$$\frac{1}{1-\frac{3}{2}\cdot\cos x} = r$$
$$1 - \frac{3}{2}\cos x = \frac{1}{r}$$
$$\frac{3}{2}\cos x = \frac{r-1}{r}$$
$$\cos x = \frac{2(r-1)}{3r}$$

$$-\frac{2}{3} < \frac{2(r-1)}{3r} < \frac{2}{3}$$
$$-1 < \frac{r-1}{r} < 1$$
$$-r < (r-1) < r$$
$$-2r < -1 < 0$$
$$r > \frac{1}{2}$$

Oppgave 8 (3 poeng)

$$\Sigma_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

P(1) :

$$\begin{aligned} 1^2 &= \frac{1(2\cdot 1-1)(2\cdot 1+1)}{3} \\ 1 &= \frac{1\cdot 1\cdot 3}{3} \\ 1 &= 1 \end{aligned}$$

P(n+1) :

$$\begin{aligned} 1^2 + 3^2 + 5^2 + ... + (2n-1)^2 + (2(n+1)-1)^2 &= \frac{n(2n-1)(2n+1)}{3} + (2(n+1)-1)^2 \\ &= \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \\ &= \frac{n(2n-1)(2n+1)}{3} + \frac{3(2n+1)^2}{3} \\ &= \frac{n(2n-1)(2n+1) + 3(2n+1)^2}{3} \\ &= \frac{(2n+1)(n(2n-1) + 3(2n+1))}{3} \\ &= \frac{(2n+1)(2n^2 - n + 6n + 3)}{3} \\ &= \frac{(2n+1)(2n^2 + 5n + 3)}{3} \end{aligned}$$

$$\begin{aligned} \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3} &= \frac{(n+1)(2n+1)(2n+3)}{3} \\ &= \frac{(2n+1)(n+1)(2n+3)}{3} \\ &= \frac{(2n+1)(2n^2+5n+3)}{3} \end{aligned}$$