Eksamen R2
Vår 2022
Løsningsforslag

Feil og mangler kan forekomme - meld fra til Marianne :)

DEL 1

Oppgave 1 (4 poeng)

a)

$$
\begin{aligned}
f(x) &= 3x \cdot \sin x \\
f'(x) &= 3 \cdot \sin x + 3x \cdot \cos x \\
&= \underline{\underline{3(\sin x + x \cdot \cos x)}}
\end{aligned}
$$

b)

$$
\begin{aligned}
g(x) &= \frac{\sin(2x)}{\cos x} \\
g'(x) &= \frac{2\cos(2x) \cdot \cos x - \sin(2x) \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{2(\cos^2 x - \sin^2 x) \cdot \cos x + 2\sin x \cos x \sin x}{\cos^2 x} \\
&= \frac{2\cos^3 x - 2\sin^2 x \cos x + 2\sin^2 x \cos x}{\cos^2 x} \\
&= \frac{2\cos^3 x}{\cos^2 x} \\
&= \underline{\underline{2\cos x}}
\end{aligned}
$$

eller :)

$$
\begin{aligned}
g(x) &= \frac{\sin(2x)}{\cos x} \\
&= \frac{2\sin x \cos x}{\cos x} \\
&= 2\sin x \\
g'(x) &= \underline{\underline{2\cos x}}
\end{aligned}
$$

Oppgave 2 (4 poeng)

a)

$$\int (e^x - \sin x)\ dx = e^x + \cos x + C$$

b)

$$\int \sin x \cdot \cos x\ dx = \sin^2 x - \int \sin x \cdot \cos x\ dx$$
$$MR: u = \sin x\ ,\ u' = \cos x$$
$$v = \sin x\ ,\ v' = \cos x$$
$$2 \cdot \int \sin x \cdot \cos x\ dx = \sin^2 x + C$$
$$\int \sin x \cdot \cos x\ dx = \frac{1}{2}\sin^2 x + C$$

$$\int_0^{\pi/4} \sin x \cdot \cos x\ dx = [\frac{1}{2}\sin^2 x]_0^{\pi/4}$$
$$= (\frac{1}{2}\sin^2 \frac{\pi}{4}) - (\frac{1}{2}\sin^2 0)$$
$$= \frac{1}{2}((\frac{\sqrt{2}}{2})^2 - 0)$$
$$= \frac{1}{2} \cdot \frac{2}{4}$$
$$= \frac{1}{4}$$

eller

$$\int \sin x \cdot \cos x\ dx = \frac{1}{2}\int \sin(2x)\ dx$$
$$MR: u = 2x\ ,\ u' = 2$$
$$= \frac{1}{4}\int \sin(u)\ du$$
$$= -\frac{1}{4}(\cos^2 x - \sin^2 x) + C$$
$$= -\frac{1}{4}(1 - 2\sin^2 x) + C$$
$$= \frac{1}{2}\sin^2 x + C$$

Oppgave 3 (2 poeng)

$$
\begin{aligned}
2 \cdot \cos(3x) &= -\sqrt{3} \\
\cos(3x) &= -\frac{\sqrt{3}}{2} \\
3x = \frac{5\pi}{6} + n \cdot 2\pi &\vee 3x = \frac{7\pi}{6} + n \cdot 2\pi \\
x = \frac{5\pi}{18} + n \cdot \frac{2\pi}{3} &\vee x = \frac{7\pi}{18} + n \cdot \frac{2\pi}{3} \\
x &= \left\{ \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18} \right\}
\end{aligned}
$$

Oppgave 4 (2 poeng)

$$
\begin{aligned}
y' + 2y &= 4 \\
y' \cdot e^{2x} + y \cdot 2e^{2x} &= 4 \cdot e^{2x} \\
(y \cdot e^{2x})' &= 4 \cdot e^{2x} \\
\int (y \cdot e^{2x})' \; dy &= 4 \int e^{2x} \; dx \\
y \cdot e^{2x} &= 4 \cdot \frac{1}{2} e^{2x} + C \\
y &= 2 + C \cdot e^{-2x} \\
\text{Initialbetingelser} : y(0) &= 1 \\
1 &= 2 + C \cdot e^{0} \\
C &= -1 \\
y &= 2 - e^{-2x}
\end{aligned}
$$

Oppgave 5 (4 poeng)

a)

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
\int \frac{1}{x}\ dx &= \ln x + C \\
\int_1^k \frac{1}{x}\ dx &= \Big[\ln x\Big]_1^k \\
&= \ln k - \ln 1 \\
&= \ln k \\
\ln k &= 2 \\
k &= e^2
\end{aligned}$$

b)

$$\begin{aligned}
V &= \pi \int_1^4 f(x)^2\ dx \\
&= \pi \int_1^4 \frac{1}{x^2}\ dx \\
&= \pi \Big[-\frac{1}{x}\Big]_1^4 \\
&= \pi\bigg(\Big(-\frac{1}{4} - (-1)\Big)\bigg) \\
&= \frac{3\pi}{4}
\end{aligned}$$

Oppgave 6 (6 poeng)

$A(2,3,-7)$, $B(-2,1,-3)$, $C(3,5,-5)$

a) For å finne likningen til planet trenger vi en normalvektor:

$$\begin{aligned}\overrightarrow{AB} &= [-2-2, 1-3, -3-(-7)] = [-4,-2,4] \\ \overrightarrow{AC} &= [3-2, 5-3, -5-(7)] = [1,2,2] \\ \overrightarrow{AB}\times\overrightarrow{AC} &= [-4-8, -(-8-4), -8-(-2)] \\ &= [-12, 12, -6] \\ &= -6[2,-2,1]\end{aligned}$$

Likningen for planet kan da skrives som :

$$\begin{aligned}2(x-2)-2(y-3)+(z+7) &= 0 \\ 2x-4-2y+6+z+7 &= 0 \\ 2x-2y+z+9 &= 0\end{aligned}$$

, som skulle vises.

b) Retningsvektoren til linja er :

$$\begin{aligned}\overrightarrow{PQ} &= [6-3, 3-1, -4-(-2)] \\ &= [3,2,-2]\end{aligned}$$

Dersom linja er parallell med planet vil $r_l \perp n_\alpha = 0$

$$\begin{aligned}r_l \cdot n_\alpha &= [3,2,-2]\cdot[2,-2,1] \\ &= 6-4-2 \\ &= 0\end{aligned}$$

Altå er linja parallell med planet, som skulle vises.

Oppgave 7 (8 poeng)

a)

$$2\cos^2 x + \sin(2x) = 0$$
$$2\cos x(\cos x + \sin x) = 0$$
$$\cos x = 0 \vee \cos x = -\sin x$$
$$x = \frac{\pi}{2} + n\cdot\pi \vee x = \frac{3\pi}{4} + n\cdot\pi$$

Da er nullpunktene :

$$x = \left\{\frac{-\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$$

b)

$$2\cos^2 x + \sin(2x) = (1 + \cos(2x)) + \sin(2x)$$

Mellomregning :

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1$$
$$2\cos^2 x = \cos(2x) + 1$$

$$2\cos^2 x + \sin(2x) = \cos(2x)) + \sin(2x) + 1$$
$$\cos(2x) + \sin(2x) = \sqrt{2}\Big(\cos(2x)\cdot\frac{\sqrt{2}}{2} + \sin(2x)\cdot\frac{\sqrt{2}}{2}\Big)$$
$$A = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$= \sqrt{2}\left(\cos(2x)\cdot\sin\left(\frac{\pi}{4}\right) + \sin(2x)\cdot\cos\left(\frac{\pi}{4}\right)\right)$$
$$= \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$$
$$2\cos^2 x + \sin(2x) = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right) + 1$$

som skulle vises.

c)
På toppen vil $\sin(2x + \frac{\pi}{4}) = 1$

$$
\begin{aligned}
\sin(2x + \frac{\pi}{4}) &= 1 \\
2x + \frac{\pi}{4} &= \frac{\pi}{2} + n \cdot 2\pi \\
2x &= \frac{\pi}{2} - \frac{\pi}{4} + n \cdot 2\pi \\
2x &= \frac{\pi}{4} + n \cdot 2\pi \\
x &= \frac{\pi}{8} + n \cdot \pi
\end{aligned}
$$

På bunnen vil $\sin(2x + \frac{\pi}{4}) = -1$

$$
\begin{aligned}
\sin(2x + \frac{\pi}{4}) &= -1 \\
2x + \frac{\pi}{4} &= \frac{3\pi}{2} + n \cdot 2\pi \\
2x &= \frac{3\pi}{2} - \frac{\pi}{4} + n \cdot 2\pi \\
2x &= \frac{5\pi}{4} + n \cdot 2\pi \\
x &= \frac{5\pi}{8} + n \cdot \pi
\end{aligned}
$$

Oppgave 8 (4 poeng)

$a_2 = 8$ og $a_4 = 2$

a)
Aritmetisk rekke : $a_n = a_1 + d(n-1)$
$d = \frac{a_4 - a_2}{4-2} = -3$

$11 + 8 + 5 + 2 - 1 - 4 - .....$

b)
Geometrisk rekke : $a_n = a_1 \cdot k^{n-1}$

$$
\begin{aligned}
a_2 &= a_1 \cdot k = 8 \\
a_1 &= \frac{8}{k} \\
a_4 &= a_1 \cdot k^3 = \frac{8}{k} \cdot k^3 = 8k^2 = 2 \\
k &= \sqrt{\frac{1}{4}} = \pm\frac{1}{2}
\end{aligned}
$$

Setter inn $k = \frac{1}{2}$:

$$
\begin{aligned}
a_1 &= \frac{8}{k} = 16 \\
a_n &= a_1 \cdot k^{n-1} = 16 \cdot \left(\frac{1}{2}\right)^{n-1} \\
&= \frac{2^4}{2^{n-1}} = 2^{4+1-n} = 2^{5-n}
\end{aligned}
$$

$16 + 8 + 4 + 2 + 1 + \frac{1}{2} + ... = 2^4 + 2^3 + 2^2 + 2^1 + 2^{-1} + ...$

$$
\begin{aligned}
s_n &= a_1 \frac{k^n - 1}{k - 1} \\
s_n &= 16 \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = 16 \cdot \frac{\frac{1}{2^n} - 1}{-\frac{1}{2}} \\
&= -32\left(\frac{1}{2^n} - 1\right) = 32\left(1 - \frac{1}{2^n}\right) \\
s_6 &= 32\left(1 - \frac{1}{2^6}\right) = 32\left(1 - \frac{1}{64}\right) \\
&= 32 \cdot \frac{64 - 1}{64} \\
&= \frac{63}{2}
\end{aligned}
$$

Setter inn $k = -\frac{1}{2}$:

$$\begin{aligned} a_1 &= \frac{8}{k} = 16 \\ a_n &= a_1 \cdot k^{n-1} = 16 \cdot \Big(-\frac{1}{2}\Big)^{n-1} \\ &= \frac{2^4}{(-2)^{n-1}} \end{aligned}$$

$$\begin{aligned} 16 - 8 + 4 - 2 + 1 - \frac{1}{2} + ... &= 2^4 - 2^3 + 2^2 - 2^1 + 2^{-1} + ... \\ s_6 &= 16 - 8 + 4 - 2 + 1 - \frac{1}{2} \\ &= 8 + 2 + \frac{1}{2} \\ &= \frac{21}{2} \end{aligned}$$

Oppgave 9 (2 poeng)

P(1) :
$a_1 = (-1)^2 \cdot 1^2 = 1$
$a_1 = (-1)^{1-1} \cdot \frac{1\cdot(1+1)}{2} = (-1)^0 \cdot \frac{2}{2} = 1$
Sann for n=1.

P(n):
$1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1} \cdot n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$
Antas sann.

P(n+1):

$$\begin{aligned}
a_{n+1} &= 1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1} \cdot n^2 + (-1)^{(n+1)+1} \cdot (n+1)^2 \\
&= (-1)^{n-1} \cdot \frac{n(n+1)}{2} + (-1)^{(n+1)+1} \cdot (n+1)^2 \\
&= (-1)^n \cdot (-1)^{-1} \cdot \frac{n(n+1)}{2} + (-1)^n \cdot (-1)^2 \cdot \frac{2(n+1)^2}{2} \\
&= (-1)^n \cdot \frac{-n(n+1) + 2(n+1)^2}{2} \\
&= (-1)^n \cdot \frac{(n+1)(2(n+1) - n)}{2} \\
&= (-1)^n \cdot \frac{(n+1)(n+2)}{2}
\end{aligned}$$

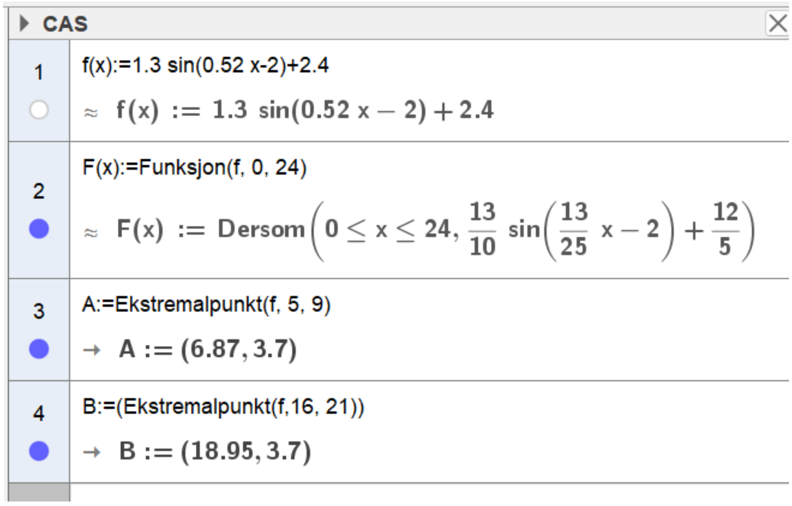
$$\begin{aligned}
a_{n+1} &= (-1)^{(n+1)-1} \cdot \frac{(n+1)((n+1)+1)}{2} \\
&= (-1)^n \cdot \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Da er påstanden sann for alle positive heltallige verdier av n.

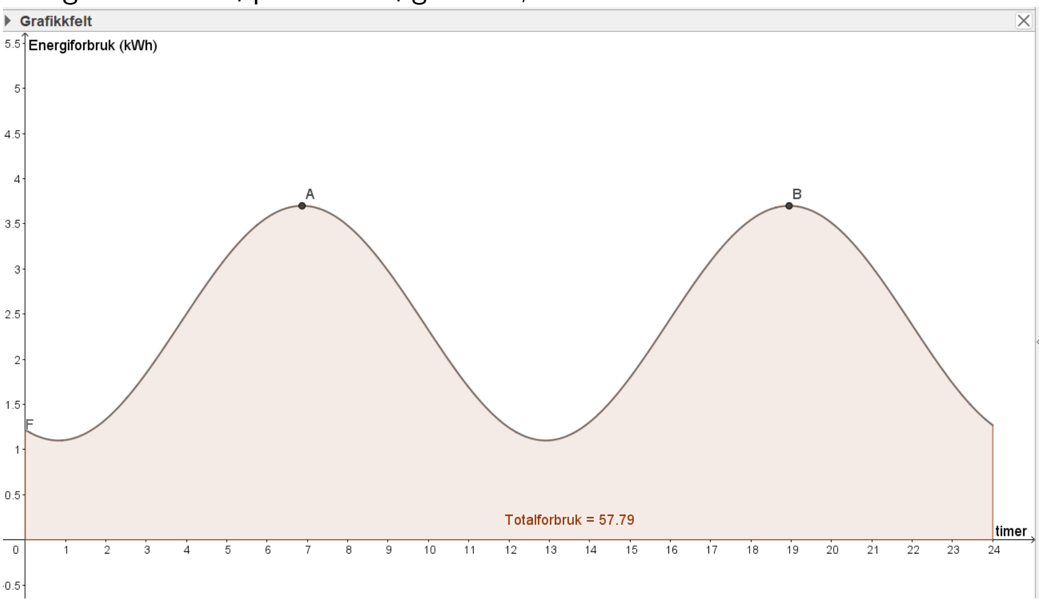
DEl 2

Oppgave 1 (6 poeng)

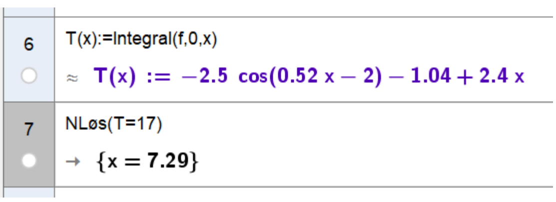
a)
Energiforbruket er størst litt før kl.07.00 (6.87 timer etter midnatt) og kl. 19.00 (18.95 timer etter midnatt).



b)
Energiforbruket i løpet av et døgn er 57,8 kWh.



c)
Det er brukt 17 kWh når klokka er ca. kl. 07.20



Oppgave 2 (6 poeng)

CAS	
1	A:=(0,7,5) → $A := (0,7,5)$
2	B:=(1,7,2) → $B := (1,7,2)$
3	C:=(-2,2,0) → $C := (-2,2,0)$
4	D:=(1,1,h) → $D := (1,1,h)$
5	Linje(A, B) → $X = (0,7,5) + \lambda\,(1,0,-3)$
6	Linje(C, D) → $X = (-2,2,0) + \lambda\,(3,-1,h)$

a)
Eksakt verdi av h slik at linjene skjærer hverandre er : $h = -\frac{56}{5}$

7	t=-2+3 s → $t = 3\,s - 2$
8	7=2-s → $7 = -s + 2$
9	5-3t=s h → $-3\,t + 5 = h\,s$
10	{$7, $8, $9} Løs: $\left\{\left\{h = \frac{-56}{5}, s = -5, t = -17\right\}\right\}$

b)
Linjen l ligger i planet dersom både A og B ligger i planet.

14	α_A:=h 7 + 3 0 + 9 7 + 5 = 7h + 68 ≈ $\alpha_A : 7\,h + 68 = 7\,h + 68$
15	α_B:=h 7 + 3 1 + 9 7 + 2 = 7h + 68 ≈ $\alpha_B : 7\,h + 68 = 7\,h + 68$

Linjen m er parallell med planet dersom den står vinkelrett på normalvektoren til planet.

16	n_α:=(3,h+9,1) → $n_\alpha := \begin{pmatrix} 3 \\ h+9 \\ 1 \end{pmatrix}$
17	r_m:=(3,-1,h) → $r_m := \begin{pmatrix} 3 \\ -1 \\ h \end{pmatrix}$
18	n_α*r_m → 0

c)

Når avstanden mellom linjene l og m er 4, vil det si at avstand fra D til planet er 4 fordi m er parallell med planet,
h blir da $h = -8.7$ eller $h = -21.6$

11	α:=3 x+(h+9) y+z=68+7 h → $\alpha : h\,y + 3\,x + 9\,y + z = 7\,h + 68$
12	Avstand(D, α) → $\sqrt{\frac{(5\,h+56)^2}{(h+9)^2+10}}$
13	Løs(sqrt((5h + 56)² / ((h + 9)² + 10))=4,h) ≈ $\{h = -21.57, h = -8.66\}$

Oppgave 3 (6 poeng)

a)
Endringen :

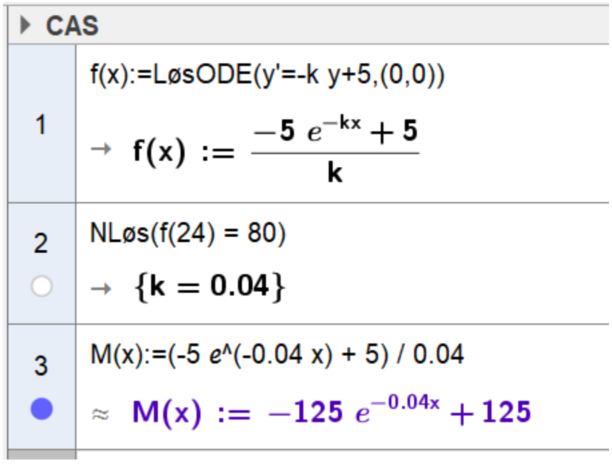
INN $= 5$	Konstant tilførsel på 5 mg / time.
UT $= k \cdot M(x)$	k=proporsjonalitetskonstant .

Dette gir en endring på $M'(x) = 5 - k \cdot M(x)$

$M(0) = 0$ - når vi starter $(t = 0)$ er medisinmengden null.

b)

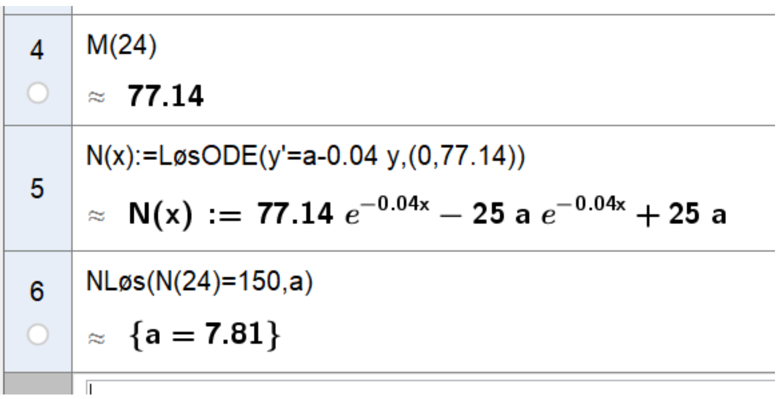
$$M(x) = .125e^{-0.04x} + 125$$



c)

Etter 24 timer vil pasienten ha 77.14 mg medisin i kroppen.

For at mengden virkestoff i blodet skal være 150 mg etter 48 timer, må vi øke dosen etter 24 timer til 7.8 mg pr.time.



Oppgave 4 (6 poeng)

a)

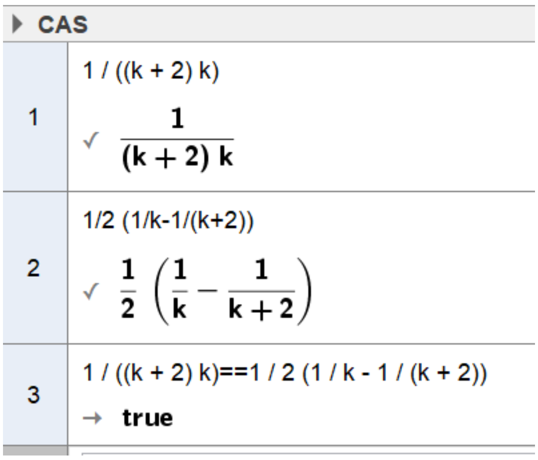
$$
\begin{aligned}
\frac{1}{k(k+2)} &= \frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)\\
&= \frac{1}{2}\left(\frac{(k+2)-k}{k(k+2)}\right)\\
&= \frac{1}{2}\left(\frac{2}{k(k+2)}\right)\\
&= \frac{1}{k(k+2)}
\end{aligned}
$$

Som skulle vises.

Vi kan også vise dette ved polynomdivisjon :

$$
\begin{aligned}
\frac{1}{k(k+2)} &= \frac{A}{k}+\frac{B}{k+2}\\
MR:\ 1 &= A(k+2)+Bk\\
k=0: 1 &= 2A \Rightarrow A=\frac{1}{2}\\
k=-2: 1 &= -2B \Rightarrow B=-\frac{1}{2}\\
&= \frac{1}{2}\cdot\frac{1}{k}-\frac{1}{2}\cdot\frac{1}{k+2}\\
&= \frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)
\end{aligned}
$$

Eller vi kan vise det i CAS:



b)

$$
\begin{aligned}
a_1 &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{1+2}\right) && = \frac{1}{2}\left(1 - \frac{1}{3}\right) \\
a_2 &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2+2}\right) && = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) \\
a_3 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{3+2}\right) && = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) \\
a_4 &= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4+2}\right) && = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) \\
a_5 &= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{5+2}\right) && = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) \\
a_6 &= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{6+2}\right) && = \frac{1}{2}\left(\frac{1}{6} - \frac{1}{8}\right) \\
s_6 &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8}\right) \\
&= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \frac{1}{7} + \cancel{\frac{1}{6}} - \frac{1}{8}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}\right)
\end{aligned}
$$

Som skulle vises.

c)

$$
\begin{aligned}
s_n = \frac{1}{2}\Bigg(&(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{6} - \frac{1}{8}) + \\
&.... + (\frac{1}{k-2} - \frac{1}{k}) + (\frac{1}{k-1} - \frac{1}{k+1}) + (\frac{1}{k} - \frac{1}{k+2})\Bigg) \\
s_n = \frac{1}{2}\Bigg(&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... + \frac{1}{k-2} + \frac{1}{k-1} + \frac{1}{k} \\
&- \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - .... - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+2}\Bigg) \\
s_n = \frac{1}{2}\Bigg(&1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\Bigg)
\end{aligned}
$$

Det vil si at den uendelige rekken har et endelig svar, altså er den konvergent.

d)
Dersom hver enkelt tilsvarende ledd i s_n er større, vil også summen av rekka være større.

$$\begin{aligned} s_n &> \sum_{k=1}^{n} \frac{1}{(k+1)^2} \\ \frac{1}{k(k+2)} &> \frac{1}{(k+1)^2} \\ k(k+2) &< (k+1)^2 \\ k^2+2k &< k^2+2k+1 \\ 0 &< 1 \end{aligned}$$

Da har vi vist at : $s_n > \sum_{k=1}^{n} \frac{1}{(k+1)^2}$

Siden summen av denne rekka er mindre enn summen av rekka i c) må den være konvergent.